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Exact eigenvalue correspondences between laminated plate theories via membrane vibration

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Abstract

Based on Reddy's third-order theory, the first-order theory and the classical theory, exact explicit eigenvalues are found for compression buckling, thermal buckling and vibration of laminated plates via analogy with membrane vibration. These results apply to symmetrically laminated composite plates with transversely isotropic laminae and simply supported polygonal edges. Comprehensive consideration of a Winkler–Pasternak elastic foundation, a hydrostatic inplane force, an initial temperature increment and rotary inertias is incorporated. Bridged by the vibrating membrane, exact correspondences are readily established between any pairs of buckling and vibration eigenvalues associated with different theories. Positive definiteness of the critical hydrostatic pressure at buckling, the thermobukling temperature increment and, in the range of either tension loading or compression loading prior to occurrence of buckling, the natural vibration frequency is proved. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Equivalent single-layer theories (Noor and Burton, 1989; Reddy and Robbins Jr, 1994; Reddy, 1997) treat a heterogeneous laminated plate as a statistically equivalent single layer, possibly having complicated constitutive behavior. Examples are the classical theory and the first-order shear deformation theory, which are based on linear distribution of the inplane displacements in the thickness direction, and the higher-order theory (Reddy, 1984) based on a nonlinear distribution of the inplane displacements in the thickness direction. The advantage of the equivalent single-layer theories by

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introducing a global displacement approximation in the thickness direction is that only 3 or 5 generalized displacement parameters are involved in the resulting equations and the order of the governing equations is independent of the total number of layers. Although the distributions of the stresses and displacements through the thickness obtained by these theories are not so accurate because transverse stresses do not satisfy continuity at layer interfaces, the global response characteristics predicted by shear deformation theories are quite accurate. The governing equations of the shear deformation theories are complicated, however, compared with the classical laminated plate theory.

This paper addresses the buckling and vibration problems of symmetrically laminated plates, with special emphasis on seeking correspondences between eigenvalues derived using Reddy's higher-order theory, the first-order theory and the classical theory. A comprehensive study presented on the issue provides exact explicit relationships via the simple solution of the membrane vibration. These correspondences apply to simply supported prestressed plates resting on a Winkler–Pasternak elastic foundation. From these exact correspondences established in the paper one can simply find exact buckling and vibration eigenvalues of polygonal laminated plates by knowing any one of these, e.g. in terms of plenty of membrane and classical plate results which can be found in any relevant books. From a technical viewpoint, this also bypasses much more complicated calculations using shear deformation laminated plate theories, instead, using relatively simple theories.

Although transversely isotropic laminae are only involved in the relationships, such composite materials have been found to have wide applications to missiles and re-entry vehicle structures (Librescu and Stein, 1992). Because of their special thermomechanical properties suited for the thermal protection of aerospace vehicles and their high flexibility in transverse shear, the present investigation appears to be of both theoretical and practical importance.

2. Governing equations

Consider a laminated plate of thickness h, resting on a Winkler–Pasternak elastic foundation. The plate consists of homogeneous, transversely isotropic laminae with uniform thickness, symmetrically disposed both from a material and geometric properties standpoint about the midplane. Let $\{x_i\}$ $(i = 1, 2, 3)$ be a Cartesian coordinate system, with the x_3-axis normal to the plane of the plate. The undeformed midplane is chosen as the reference plane defined by $x_3 = 0$.

Throughout the following derivations, a comma followed by a subscript denotes a derivative with respect to the corresponding spatial coordinate. The Einsteinian summation convention applies, unless specified otherwise, to repeated subscripts of tensor components, with Latain subscripts ranging from 1 to 3 while Greek subscripts are either 1 or 2.

Reddy's (1984) third-order theory for symmetric laminates is based on the following displacement field:

$$v_\alpha(x_i;\, t) = -x_3 u_{3,\alpha} + g\varphi_\alpha, \quad v_3(x_i;\, t) = u_3, \tag{1}$$

where the deflection u_3 and the generalized displacement φ_α are independent of x_3, and

$$g(x_3) = x_3\left(1 - \frac{4x_3^2}{3h^2}\right), \quad \varphi_\alpha = u_{3,\alpha} + \psi_\alpha. \tag{2a,b}$$

The displacement field (1) is essentially the same as the one in Reddy's (1984) original work in the case of symmetric laminates, where the function ψ_α was used through a substitution of eqn (2b).

For a laminated plate subjected to inplane hydrostatic pressure N per unit length exerted on edges, the steady-state linear governing equations with a time-harmonic dependence $\exp(i\omega t)$ are expressed as

$$M_{\alpha\beta,\alpha\beta} - Nu_{3,\alpha\alpha} - ku_3 + Gu_{3,\alpha\alpha} + I_0\omega^2 u_3 - I_1\omega^2 u_{3,\alpha\alpha} + I_2\omega^2\varphi_{\alpha,\alpha} = 0, \tag{3}$$

$$P_{\alpha\beta,\beta} - R_\alpha - I_2\omega^2 u_{3,\alpha} + I_3\omega^2\varphi_\alpha = 0, \tag{4}$$

where ω denotes an angular frequency, k and G denote the Winkler–Pasternak foundation parameters (Kerr, 1964), and

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} x_3 \, dx_3, \quad P_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} g \, dx_3, \quad R_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3} g_{,3} \, dx_3, \tag{5}$$

$$\sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho} e_{\omega\rho}, \quad \sigma_{\alpha 3} = 2E_{\alpha 3\omega 3} e_{\omega 3}, \quad e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \tag{6}$$

$$[I_0, I_1, I_2, I_3] = \int_{-h/2}^{h/2} \rho\left[1, x_3^2, x_3 g, g^2\right] dx_3. \tag{7}$$

In eqns (3)–(6) as well as in what follows, the time-harmonic factor $\exp(i\omega t)$ has been omitted and each physical quantity refers to its spatial part.

The components of the elasticity tensor associated with a transversely isotropic material, with its isotropy plane being parallel to the reference plane, are expressed as (Librescu, 1975)

$$H_{\alpha\beta\omega\rho} = \frac{vE}{1 - v^2}\delta_{\alpha\beta}\delta_{\omega\rho} + \frac{E}{2(1+v)}\left(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}\right), \quad E_{\alpha 3\omega 3} = \mu'\delta_{\alpha\omega}, \tag{8}$$

where E and v denote Young's modulus and Poisson's ratio in the plane of isotropy, respectively, and μ' denotes the shear modulus in the plane normal to the isotropy plane.

Through eqns (6) and (8) and the spatial counterpart of eqns (1), eqn (5) may be written in an alternative form as

$$\begin{bmatrix} M_{\alpha\beta} \\ P_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} -a_1 + b_1 & a_2 - b_2 \\ -a_2 + b_2 & a_3 - b_3 \end{bmatrix} \begin{bmatrix} u_{3,\omega\omega} \\ \varphi_{\omega,\omega} \end{bmatrix} \delta_{\alpha\beta} + \begin{bmatrix} -b_1 & b_2 \\ -b_2 & b_3 \end{bmatrix} \begin{bmatrix} u_{3,\alpha\beta} \\ \frac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) \end{bmatrix}, \quad R_\alpha = c\varphi_\alpha, \tag{9a,b}$$

where

$$[a_1, a_2, a_3] = \int_{-h/2}^{h/2} \left[x_3^2, x_3 g, g^2\right] \frac{E}{1 - v^2} \, dx_3,$$

$$[b_1, b_2, b_3] = \int_{-h/2}^{h/2} \left[x_3^2, x_3 g, g^2\right] \frac{E}{1 + v} \, dx_3,$$

$$c = \int_{-h/2}^{h/2} \mu'(g_{,3})^2 \, dx_3. \tag{10a–c}$$

With the expressions of eqn (9), the governing eqns (3) and (4) are then expressed in terms of three displacement functions u_3 and φ_α as

$$-a_1u_{3,\,\alpha\alpha\beta\beta}+a_2\varphi_{\alpha,\alpha\beta\beta}-Nu_{3,\alpha\alpha}-ku_3+Gu_{3,\alpha\alpha}+I_0\omega^2u_3-I_1\omega^2u_{3,\alpha\alpha}+I_2\omega^2\varphi_{\alpha,\alpha}=0, \tag{11}$$

$$-a_2u_{3,\,\alpha\beta\beta}+\frac{1}{2}b_3\varphi_{\alpha,\beta\beta}+\left(a_3-\frac{1}{2}b_3\right)\varphi_{\beta,\beta\alpha}-c\varphi_\alpha-I_2\omega^2u_{3,\alpha}+I_3\omega^2\varphi_\alpha=0. \tag{12}$$

Alternatively, the following matrix equation can be obtained through eqn (11) and differentiating eqn (12) with respect to x_α,

$$\mathbf{KX}=\mathbf{0}, \tag{13}$$

where $\mathbf{X}=[u_3\quad\varphi_{\alpha,\alpha}]^{\mathrm{T}}$, $\mathbf{0}=[0\quad 0]^{\mathrm{T}}$ and $\mathbf{K}=(K_{IJ})$ is a 2×2 operator matrix in which its elements, expressed in terms of two-dimensional Laplacian operator ∇^2, are

$$K_{11}(\nabla^2)=-a_1\nabla^4-\left(N-G+I_1\omega^2\right)\nabla^2-k+I_0\omega^2,$$

$$K_{12}(\nabla^2)=a_2\nabla^2+I_2\omega^2,$$

$$K_{21}(\nabla^2)=-a_2\nabla^4-I_2\omega^2\nabla^2,$$

$$K_{22}(\nabla^2)=a_3\nabla^2-c+I_3\omega^2. \tag{14}$$

Furthermore, eliminating $\varphi_{\alpha,\alpha}$ from eqn (13) gives

$$\det\left[\mathbf{K}(\nabla^2)\right]u_3=\left(a_2^2-a_1a_3\right)\left(\nabla^2+\lambda_1\right)\left(\nabla^2+\lambda_2\right)\left(\nabla^2+\lambda_3\right)u_3=0, \tag{15}$$

where λ_I ($I=1, 2, 3$) are three roots of the cubic equation

$$\det\left[\mathbf{K}(-\lambda)\right]=K_{11}(-\lambda)K_{22}(-\lambda)-K_{12}(-\lambda)K_{21}(-\lambda)=0. \tag{16}$$

Eqn (15) is the characteristic equation, from which the eigenvalues and associated eigenfunctions for buckling and vibration problems of Reddy's third-order theory can be solved with given boundary conditions.

3. Simply supported edges of polygonal plates

Assuming that a laminated plate is simply supported on its boundary, the boundary condition is expressed as

$$u_3=0,\quad P_{\mathrm{NN}}=0,\quad \varphi_{\mathrm{T}}=0,\quad M_{\mathrm{NN}}=0,\quad u_{3,\mathrm{T}}=0, \tag{17a–e}$$

where the upper case subscripts N and T denote the directions normal and tangential to the boundary, respectively. No implicit summation applies to the repeated upper case subscripts.

For a polygonal laminated plate with rectilinear edges, eqn (17e) is identically satisfied due to eqn (17a), while eqns (17b, d) can be recast as, through eqns (19a) and (17a, c),

$$u_{3,\mathrm{NN}}=0,\quad \varphi_{\mathrm{N,N}}=0. \tag{18}$$

Therefore, the condition of simply supported straight edges can be expressed as

$$u_3 = 0, \quad \nabla^2 u_3 = 0, \quad \varphi_{\mathrm{T}} = 0, \quad \varphi_{\alpha,\,\alpha} = 0. \tag{19a–d}$$

As a consequence of eqns (13) and (19),

$$\nabla^4 u_3 = 0, \quad \nabla^2 \varphi_{\alpha,\,\alpha} = 0. \tag{20a,b}$$

Note that eqns (20) are not independent restraint conditions.

4. Analogy between Reddy and membrane theories

In order to facilitate subsequent analysis, eqn (15) may be written in an alternative form as

$$\left(\nabla^2 + \lambda_{I_1}\right)F_{I_1} = 0, \quad F_{I_1} \equiv \left(a_2^2 - a_1 a_3\right)\left(\nabla^2 + \lambda_{I_2}\right)\left(\nabla^2 + \lambda_{I_3}\right)u_3, \tag{21a,b}$$

where λ_{I_1} is one of the three roots of the cubic eqn (16), while λ_{I_2} and λ_{I_3} are the other two. No implicit summation applies to the repeated subscripts in eqn (21a). In view of eqns (19a, b), (20a) and (21b), the Helmholtz eqn (21a) is shown to be associated with such a boundary condition as

$$F_{I_1} = 0. \tag{22}$$

According to the properties of a cubic equation with real coefficients, there exists at least a real root of the cubic equation, while the other two are either real or complex conjugate. Taking λ_{I_1} as a real root, then the operator $(\nabla^2 + \lambda_{I_2})\,(\nabla^2 + \lambda_{I_3})$ must be real no matter whether λ_{I_2} and λ_{I_3} are real or complex conjugate roots. For a practical physical system, only the real displacement function u_3 is of interest. In view of eqn (21b), it can be concluded that F_{I_1} is a real eigenfunction, related to a physically reasonable u_3. Therefore, the eigenvalue problem for Reddy's third-order laminated plate theory is composed of eqn (21a) and the boundary condition (22), which is a boundary value problem of Dirichlet type. This boundary value problem is analogous to a uniform membrane executing small transverse vibration. Insofar as the case of three real roots is concerned, λ_{I_1} has not been specified as a particular root of the three real roots, thus the general case has been considered in the present analysis.

The eigenvalue of the membrane vibration problem is known as (Gladwell and Willms, 1995)

$$\lambda_{\mathrm{M}} = \frac{\rho_{\mathrm{M}}\omega_{\mathrm{M}}^2}{S}, \tag{23}$$

with ρ_{M}, S and ω_{M} being the mass density, constant tension and vibration frequency of the membrane, respectively.

It is obvious that the eigenvalue λ of the Dirichlet boundary value problem, eqns (21a) and (22), for the Reddy plate theory must be the same as λ_{M}, i.e.

$$\lambda = \lambda_{\mathrm{M}}. \tag{24}$$

Since λ is a real root of the cubic eqn (16), substituting eqn (24) into (16) yields

$$\det\left[\mathbf{K}(-\lambda_{\mathrm{M}})\right] = A\omega^4 + B\omega^2 + C = 0, \tag{25}$$

where

$$A = \left(I_1 I_3 - I_2^2\right)\lambda_{\mathrm{M}} + I_0 I_3,$$

$$B = (2a_2I_2 - a_1I_3 - a_3I_1)\lambda_M^2 + (NI_3 - GI_3 - a_3I_0 - cI_1)\lambda_M - (kI_3 + cI_0),$$

$$C = \left(a_1a_3 - a_2^2\right)\lambda_M^2 + (ca_1 - Na_3 + Ga_3)\lambda_M^2 + (ka_3 - Nc + Gc)\lambda_M + kc. \tag{26}$$

For the buckling problem using the Reddy plate theory, the critical hydrostatic pressure can be obtained by setting $\omega = 0$ in eqn (25), i.e. $C = 0$, which gives

$$N^{\mathrm{cr}} = a_1\lambda_M + G + \frac{k}{\lambda_M} - \frac{a_2^2\lambda_M^2}{a_3\lambda_M + c}. \tag{27}$$

In particular, for a thermal buckling problem, the equivalent inplane thermal load is expressed as

$$N^{\Delta T} = \gamma\Delta T, \quad \gamma = \int_{-h/2}^{h/2} \frac{E\alpha}{1 - v}\,\mathrm{d}x_3, \tag{28}$$

where α and ΔT denote, respectively, the linear expansion coefficient of the laminated plate and a uniform temperature increment within it. Note that α is a function of the thickness coordinate. Therefore, the critical thermobuckling temperature increment ΔT^{cr} is expressed in terms of the eigenvalue of membrane vibration as

$$\Delta T^{\mathrm{cr}} = \frac{1}{\gamma}\left[a_1\lambda_M + G + \frac{k}{\lambda_M} - \frac{a_2^2\lambda_M^2}{a_3\lambda_M + c}\right]. \tag{29}$$

For the free vibration problem using the Reddy laminated plate theory, the characteristic frequency of a laminated plate subjected to an initial hydrostatic inplane stress resultant N is explicitly obtained from eqn (25) as

$$\omega^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{30}$$

Note that the stress resultant N can also be an inplane thermal load due to an initial temperature increment or the sum of both.

According to the work of Irschik (1985), there are three different types of motion for the polygonal simply supported plates. The first two of these eigenmotions, called flexural and thickness-shear modes, are independently generated by Dirichlet's boundary value problem, whereas the third mode, i.e. thickness-twist mode, is due to Neumann's boundary value problem. The analogy of eqn (24) only corresponds to the membrane with fixed edges. Correspondingly, the eigenvectors associated with the vibration frequencies given by eqn (30) exhibit flexural and thickness-shear modes.

5. The first-order theory

While taking $g(x_3) = x_3$, it can be seen from eqn (1) that the displacement field is essentially the one for the first-order plate theory (Reddy, 1997). In this case, $a_2 = a_3 = a_1$ and $I_2 = I_3 = I_1$ from eqns (10a) and (7). As is well-recognized, the shear correction factor κ should be involved in the first-order plate theory, i.e. the parameter c expressed in eqn (10c) is replaced by

$$c_F = \kappa\int_{-h/2}^{h/2} \mu'\,\mathrm{d}x_3. \tag{31}$$

The characteristic equation for the first-order laminated plate theory is the same as the matrix eqn (13), which if eliminating $\varphi_{\alpha,\alpha}$ reduces to a quadratic equation with respect to the Laplacian operator ∇^2, i.e. the degenerated form from eqn (15) due to $a_2 = a_3 = a_1$,

$$-a_1(N - G - c_F)\left(\nabla^2 + \lambda_1\right)\left(\nabla^2 + \lambda_2\right)u_3 = 0. \tag{32}$$

The boundary condition for simply supported edges is expressed as eqns (17a, c, d). The condition $M_{NN} = 0$ in eqn (17d) gives the relationship $u_{3,NN} - \varphi_{N,N} = 0$ which is different from the case of Reddy's third-order theory. However, after scrutiny of this condition with the help of eqn (13), it can be shown that the boundary condition of the first-order theory for simply supported rectilinear edges is the same as eqn (19) while $N \neq G + c_F$. In the case of $N = G + c_F$, eqn (32) is further degenerated to a linear equation with respect to the Laplacian operator ∇^2. Therefore, only the boundary condition (19a) is needed, though the boundary condition (19b) may also be further shown to hold for the particular case.

Based on the above, the results can be given as follows:

1. the critical hydrostatic inplane pressure is

$$N_F^{cr} = a_1\lambda_M + G + \frac{k}{\lambda_M} - \frac{a_1^2\lambda_M^2}{a_1\lambda_M + c_F}; \tag{33}$$

2. the critical thermobuckling temperature increment is

$$\Delta T_F^{cr} = \frac{1}{\gamma}\left[a_1\lambda_M + G + \frac{k}{\lambda_M} - \frac{a_1^2\lambda_M^2}{a_1\lambda_M + c_F}\right]; \tag{34}$$

3. the free vibration frequency of prestressed laminated plates is expressed in the same form as eqn (30) where, instead of eqn (26),

$$A = I_0 I_1,$$

$$B = (NI_1 - GI_1 - a_1 I_0 - c_F I_1)\lambda_M - (kI_1 + c_F I_0),$$

$$C = (c_F a_1 - Na_1 + Ga_1)\lambda_M^2 + (ka_1 - Nc_F + Gc_F)\lambda_M + kc_F. \tag{35}$$

6. The classical theory

In the case of the classical Kirchhoff theory for laminated plates (Reddy, 1997), eqns (1) exactly represent the corresponding displacement field if taking $g(x_3) = 0$, which yields $a_2 = a_3 = 0$ and $I_2 = I_3 = 0$ from eqns (10a) and (7). The characteristic equation is $[K_{11}(\nabla^2)]u_3 = 0$, associated with the boundary condition shown in eqns (19a, b). Similarly, therefore, the eigenvalue equation is

$$K_{11}(-\lambda_M) = 0, \tag{36}$$

from which the results can be given as follows:

1. the critical hydrostatic inplane pressure at buckling is

$$N_{\mathrm{K}}^{\mathrm{cr}}=a_1\lambda_{\mathrm{M}}+G+\frac{k}{\lambda_{\mathrm{M}}}; \tag{37}$$

2. the critical thermobuckling temperature increment is

$$\Delta T_{\mathrm{K}}^{\mathrm{cr}}=\frac{1}{\lambda}\left[a_1\lambda_{\mathrm{M}}+G+\frac{k}{\lambda_{\mathrm{M}}}\right]; \tag{38}$$

3. the free vibration frequency is

$$\omega_{\mathrm{K}}^2=\frac{a_1\lambda_{\mathrm{M}}^2-(N-G)\lambda_{\mathrm{M}}+k}{I_1\lambda_{\mathrm{M}}+I_0}. \tag{39}$$

7. Discussion on correspondences

In view of eqns (27), (33) and (37), the exact critical buckling values of the inplane hydrostatic pressure have been obtained via the membrane vibration frequency. Therefore, exact correspondences between the buckling pressures of Reddy's third-order theory, the first-order theory and the classical theory for symmetrically laminated polygonal plates with transversely isotropic laminae and simply supported rectilinear edges can be explicitly established upon substitution of λ_{M} between these expressions. Similarly, exact relationships can also be given between the thermobuckling temperature increments shown in eqns (29), (34) and (38). The same is true for the vibration frequencies expressed in eqns (30) and (39) associated with different theories. Exact explicit correspondences between buckling and vibration eigenvalues can also be found in the same way. Moreover, it is easy to provide any pairs of exact explicit relationship upon substitution of λ_{M}, although these correspondences are not explicitly shown herein.

When rotary inertias are omitted, i.e. $I_1 = I_2 = I_3 = 0$, eqn (25) furnishes the natural frequency of Reddy's theory neglecting rotary inertias

$$\hat{\omega}^2=\frac{\left(a_1a_3-a_2^2\right)\lambda_{\mathrm{M}}^3+(ca_1-Na_3+Ga_3)\lambda_{\mathrm{M}}^2+(ka_3-Nc+Gc)\lambda_{\mathrm{M}}+kc}{I_0(a_3\lambda_{\mathrm{M}}+c)} \tag{40}$$

This expression is also valid for the first-order theory by taking $a_2 = a_3 = a_1$, $c = c_{\mathrm{F}}$ and for the classical theory by taking $a_2 = a_3 = 0$, respectively, i.e.

$$\hat{\omega}_{\mathrm{F}}^2=\frac{(c_{\mathrm{F}}-N+G)a_1\lambda_{\mathrm{M}}^2+(ka_1-Nc_{\mathrm{F}}+Gc_{\mathrm{F}})\lambda_{\mathrm{M}}+kc_{\mathrm{F}}}{I_0(a_1\lambda_{\mathrm{M}}+c_{\mathrm{F}})}, \tag{41}$$

$$\hat{\omega}_{\mathrm{K}}^2=\frac{a_1\lambda_{\mathrm{M}}^2-(N-G)\lambda_{\mathrm{M}}+k}{I_0}. \tag{42}$$

Note that eqn (40) and (41) give only one value of the natural frequency, rather than two values in eqns (30) for the Reddy and first-order laminated plate theories. This implies that an additional frequency expressed in eqn (30) results from incorporating the rotary inertia. Insofar as flexural vibration is concerned, only the lower value in eqn (30) is related as normally rotational vibration within plates has a higher frequency than that of transverse vibration for plate packages.

Comparing eqns (40)–(42), the natural frequencies corresponding to Reddy's third-order theory, the first-order theory and the classical Kirchhoff theory for laminated plates excluding the effect of rotary inertias are connected with each other as follows:

$$\hat{\omega}^2 + \frac{a_2^2\lambda_M^3}{I_0(a_3\lambda_M + c)} = \hat{\omega}_F^2 + \frac{a_1^2\lambda_M^3}{I_0(a_1\lambda_M + c_F)} = \hat{\omega}_K^2. \tag{43}$$

Similarly, by comparing eqns (27), (33) and (37), the connection of the critical buckling hydrostatic pressures associated with different laminated plate theories is

$$N^{cr} + \frac{a_2^2\lambda_M^2}{a_3\lambda_M + c} = N_F^{cr} + \frac{a_1^2\lambda_M^2}{a_1\lambda_M + c_F} = N_K^{cr}. \tag{44}$$

By comparing eqns (29), (34) and (38), the critical thermobuckling temperature increments associated with different laminated plate theories are connected by

$$\Delta T^{cr} + \frac{a_2^2\lambda_M^2}{\gamma(a_3\lambda_M + c)} = \Delta T_F^{cr} + \frac{a_1^2\lambda_M^2}{\gamma(a_1\lambda_M + c_F)} = \Delta T_K^{cr}. \tag{45}$$

Therefore, the differences of the natural frequencies, the buckling loads and the thermobuckling temperature increments between different theories are clearly shown in eqns (43)–(45). It is also clear from these equations that the classical Kirchhoff laminated plate theory always overpredicts the eigenvalues, compared with Reddy's theory and the first-order theory.

In the special case of simply-supported polygonal single-layer isotropic plates, some available results, i.e. the relationships regarding critical hydrostatic inplane pressures at buckling and the natural frequencies (Conway and Farnham, 1965; Roberts, 1971; Irschik, 1985; Wang and Reddy, 1997; Wang et al., 1997) associated with different single-layer isotropic plate theories can be recovered from the present results.

In the work of Cheng et al. (1993a, b, 1994), it has been shown that the dimensionless governing equations and boundary conditions have the same form for the sandwich plate theory (Reissner, 1948, 1950) as for the first-order shear deformation theory (Reissner, 1945, 1985; Mindlin, 1951) for single-layer plates. Therefore, the correspondences with the Reissner–Mindlin single-layer plate theory also apply to the Reissner sandwich plate theory (Wang, 1995, 1996), where a_1, and I_1 for single-layer plates are replaced by the flexural rigidity and rotary inertia of sandwich plates, and $c_F = \mu_c h$ with μ_c being the shear modulus of the sandwich core.

8. Positive definiteness of eigenvalues

This paper only consider linear eigenvalue problems. Therefore, the condition

$$-\infty < N \leqslant N^{cr} \tag{46}$$

is used since $N > N^{cr}$ corresponds to a nonlinear problem, i.e. postbuckling of laminated plates. A negative value of N implies an initial inplane tension.

Based on Green's formula, it can be proved that an eigenvalue problem of the Dirichlet type contains a denumerably infinite sequence of discrete positive eigenvalues corresponding to nontrivial real eigenfunctions (Courant and Hilbert, 1953). Therefore, the following restraint condition for the eigenvalue of the membrane vibration should be used throughout this paper

$$\lambda_M > 0. \tag{47}$$

In accordance with Schwarz's integral inequality, it can be shown that

$$a_1 a_3 \geqslant a_2^2, \quad I_1 I_3 \geqslant I_2^2, \tag{48a,b}$$

with equality holding if and only if x_3 and $g(x_3)$ in the integrands involved in eqns (10a) and (7) are linearly dependent functions, i.e. corresponding to the case of the first-order plate theory. The equality also holds in the case of the classical plate theory because of $g(x_3) = 0$.

Using the inequalities (47) and (48a), it is easily seen from eqns (27) and (29) that

$$N^{\mathrm{cr}} > 0, \quad \Delta T^{\mathrm{cr}} > 0. \tag{49}$$

Since the results at mechanical buckling and thermal buckling for the first-order theory and the classical theory can be obtained from eqns (27) and (29) by setting $a_2 = a_3 = a_1$, $c = c_F$ and $a_2 = a_3 = 0$, respectively, the resulting inequalities (49) are also valid for the two theories. Thus, it has been shown that the critical hydrostatic loads at buckling and the thermobuckling temperature increments for the three laminated plate theories are positive definite.

For convenience of subsequent analysis, if denoting

$$c_1 = a_3\lambda_M + c, \quad c_2 = a_1\lambda_M^2 - (N - G)\lambda_M + k, \tag{50a,b}$$

eqns (39) and (40) may be rewritten as

$$\omega_K^2 = \frac{c_2}{I_1\lambda_M + I_0}, \quad \hat{\omega}^2 = \frac{c_1 c_2 - a_2^2\lambda_M^3}{I_0 c_1} \tag{51a,b}$$

which are, respectively, the natural frequencies of the classical theory and, in the absence of rotary inertias, of Reddy's theory.

Using eqn (27), the restraint condition $N \leqslant N^{\mathrm{cr}}$ given in eqn (46) can be recast as

$$c_1 c_2 \geqslant a_2^2\lambda_M^3. \tag{52}$$

Particularly, when $N = N^{\mathrm{cr}}$, the equality in (52) holds valid. This leads to vanishing ω_K and $\hat{\omega}$ as can be seen from eqn (51), i.e. trivial solutions. The condition $N < N^{\mathrm{cr}}$ corresponding to nontrivial solutions of vibration problems furnishes $c_1 c_2 > a_2^2\lambda_M^3$, which implies that ω_K and $\hat{\omega}$ in eqn (51) are positive definite. Note that the expression of eqn (51b) also includes the natural frequencies of the first-order and the classical theories as special cases in the limits described in the foregoing.

Finally, only the natural frequencies expressed in eqn (30), associated with the Reddy and first-order laminated theories with inclusion of rotary inertias, remain to be discussed. With the expressions of eqns (50), the coefficients A, B and C in eqn (26) may be written in an alternative form

$$A = \left(I_1 I_3 - I_2^2\right)\lambda_M + I_0 I_3,$$

$$B = -(I_1\lambda_M + I_0)c_1 - I_3 c_2 + 2a_2 I_2\lambda_M^2,$$

$$C = c_1 c_2 - a_2^2\lambda_M^3. \tag{53a–c}$$

As the natural frequencies expressed by eqn (30) are for both Reddy's theory and the first-order theory, a unified discussion on the coefficients A, B and C in eqns (53) is given. The case of the first-order theory can be achieved by taking $a_2 = a_3 = a_1$, $I_2 = I_3 = I_1$ and $c = c_F$ in eqns (53) and (50a).

With the help of the inequalities (48b) and (52), it is simply seen from eqns (53a, c) that

$$A > 0, \quad C \geqslant 0, \tag{54}$$

and, from eqn (53b),

$$B \leqslant -2\sqrt{(I_1\lambda_M + I_0)I_3c_1c_2} + 2a_2I_2\lambda_M^2 < 0. \tag{55}$$

Moreover, with the aid of the inequality (55), the discriminant of eqn (30) gives

$$B^2 - 4AC \geqslant 4\left(\sqrt{(I_1\lambda_M + I_0)I_3c_1c_2} - a_2I_2\lambda_M^2\right)^2 - 4\left[\left(I_1I_3 - I_2^2\right)\lambda_M + I_0I_3\right]\left(c_1c_2 - a_2^2\lambda_M^3\right)$$

$$= 4\lambda_M\left[\sqrt{(I_1\lambda_M + I_0)I_3}a_2\lambda_M - I_2\sqrt{c_1c_2}\right]^2 \geqslant 0. \tag{56}$$

On the basis of the inequalities (54)–(56), it is concluded that ω^2 shown in eqn (30) has two positive roots with the only exception where $C = 0$. In the case of $C = 0$, i.e. $N = N^{cr}$, eqn (30) gives two roots $\omega^2 = 0, -B/A$. The vanishing frequency is the lower one and hence related dominantly to the transverse flexural mode of a laminated plate rather than the rotational mode, corresponding to a trivial solution to the more interested eigenvalue problem of flexural vibrating plates. For $N < N^{cr}$, however, both of the natural frequencies shown in eqn (30) are positive definite.

It should be noted that for an arbitrarily given positive value of ω^2, conversely, the characteristic eqn (16) might not identically provide a positive eigenvalue λ. Therefore, while expressing λ_M in terms of ω^2 from eqn (16) or (25), the positive definiteness condition shown in the inequality (47) places a restriction on the range of ω^2.

9. Conclusions

Based on the rigorous analysis presented in this paper, exact eigenvalue correspondences between Reddy's third-order theory, the first-order theory and the classical Kirchhoff theory are established through the membrane vibration frequency. These exact explicit relationships are valid for symmetrically laminated polygonal plates with transversely isotropic laminae and simply supported rectilinear edges, under a hydrostatic inplane force and resting on a Winkler–Pasternak elastic foundation. Once an eigenvalue is given, the eigenvalues for any other problems can be obtained through the connections readily established in the paper. Some available analogies concerning single-layer and sandwich plates are special cases of the present results. Positive definiteness of the critical hydrostatic pressures, the thermobuckling temperature increments and the natural vibration frequencies is proved for the third-order, first-order and classical laminated plate theories, subject to the condition that the inplane load is either tension or compression less than the critical buckling hydrostatic pressure, i.e. prior to occurrence of buckling.

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